Cohomology Groups of the Dual Steenrod Algebra

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Abstract Algebra Fundamentals

Groups

A **group** is a set of elements with an "addition" operation +

- + is associative, with an identity element and inverses

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Example

Integers $\mathbb{Z}$ with standard + form a group

- 0 is the identity element and inverses are additive inverses

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Rings

A **ring** can be thought of as a group with an additional operation and more restrictions

- $+$ now commutative, also a "multiplication" operation $\times$
- $\times$ must be associative, distribute over $+$, and have an identity

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Example

Set of 2 by 2 matrices with real entries $\mathcal{M}_2(\mathbb{R})$ is a ring, with addition and matrix multiplication

- Multiplication is not commutative, multiplicative inverse doesn't always exist
- $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$
- $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{12} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$

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- Commutative multiplication, a multiplicative identity, and multiplicative inverses (except for the addition identity)

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Example

Set of integers modulo p (a prime) with typical addition and multiplication form a field $\mathbb{F}_p$

- $2^{-1} \equiv 4 \pmod 7$
- 0 does not have multiplicative inverse

Algebras over Fields

Algebras

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Example

The complex numbers are an algebra over the reals

- $(a+bi)+(c+di)=(a+b)+(c+d)i$
- $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$
- $r(a+bi)=(ra)+(rb)i$, for $r\in\mathbb{R}$

Modules

Modules

A **module** is like an algebra. However, modules scale over rings instead of fields, and do not require the bilinear product.

modulo 2 dual Steenrod algebra

- Consider the Steenrod algebra given by $p = 2$ (from algebraic topology)
- Obtain **dual algebra** by considering linear maps from the algebra to the field it is considered over, or $\mathbb{Z}/2$
- The modulo 2 **dual Steenrod Algebra**, denoted by $\mathcal{A}_*$, is a **polynomial ring** (Milnor)

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dual Steenrod Algebra

$\mathcal{A}_* = \mathbb{Z}/2[\xi_1, \xi_2, \ldots]$ where each ξ_i has degree $2^i - 1$

Action of $\mathbb{Z}/2$ on $\mathcal{A}_*$

- Denote nontrivial element of $\mathbb{Z}/2$ by χ
- Define the canonical **conjugation action** of $\mathbb{Z}/2$ on $\mathcal{A}_*$ inductively:

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- χ respects multiplication and addition:
  - $\chi(ab) = \chi(a)\chi(b)$
  - $\chi(a+b) = \chi(a) + \chi(b)$

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- χ respects multiplication and addition:
  - $\chi(ab) = \chi(a)\chi(b)$
  - $\chi(a + b) = \chi(a) + \chi(b)$
- χ preserves degree
- $\chi^2 = 0$

Action of $\mathbb{Z}/2$ on $\mathcal{A}_*$

Some small values of $\chi(\xi_i)$: (Note that $\xi_0 = 1$)

Computed Values

$$\begin{aligned}
\chi(1) &= 1 \\
\chi(\xi_1) &= \xi_1 \\
\chi(\xi_2) &= \xi_2 + \xi_1^3 \\
\chi(\xi_3) &= \xi_3 + \xi_1\xi_2^2 + \xi_1^4\xi_2 + \xi_1^7 \\
\chi(\xi_4) &= \xi_4 + \xi_1\xi_3^2 + \xi_1^8\xi_3 + \xi_2^5 + \xi_1^3\xi_2^4 + \xi_1^9\xi_2^2 + \xi_1^{12}\xi_2 + \xi_1^{15}
\end{aligned}$$

Homogenous: deg $\xi_2 = 2^2 - 1 = 3$, deg $\xi_1^3 = 3(2^1 - 1) = 3$

Homological Algebra

- Associate sequences of algebraic objects with other algebraic objects
- Example: Homology groups $H_n(X)$
  - Elucidate information about "holes" in topological spaces

Group Cohomology

- View $\mathcal{A}_*$ as a module over $\mathbb{Z}/2$
- Let $\mathbb{Z}/2$ act on $\mathcal{A}_*$ by the conjugation action
- The **cohomology groups** $H^n(G; M)$ of a module M and its G-action elucidate information about the action
- Goal: compute cohomology groups of this action

Fixed points

- Zeroth cohomology group $H^0(\mathbb{Z}/2, \mathcal{A}_*)$ is subalgebra A_*^χ invariant under the action χ
- We would like to compute these fixed points of χ

Summary of work

- Some elements are clearly invariant under χ:

Invariant Elements

- 1, ξ_1
- $\epsilon\chi(\epsilon)$ for any ϵ
- $\chi(\epsilon) + \epsilon$ for any ϵ

- However, these elements do not span the fixed point space!
  - Example: $\xi_2^3 + \xi_3\xi_1^2 + \xi_1^9$ in degree 9

Summary of Work

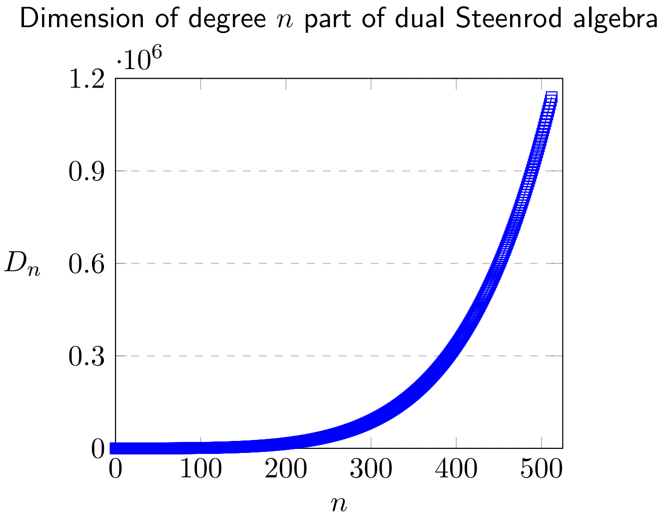
- Let D_n be the dimension of $\mathcal{A}_*$ in degree n
- Crossley and Whitehouse bounded the dimension d_n of $\mathcal{A}_*^{\chi}$ in degree n as

Bound on d_n

$$\frac{D_n}{2} \leq d_n \leq D_n - \frac{D_{n-1}}{2}$$

- Understanding D_n gives strong bounds on d_n

Summary of Work



- D_n grows faster than polynomial, but still infra-exponential

Summary of Work

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Asymptotic Behavior

$$D_n \sim \exp\left[\frac{\ln^2 n}{2\ln 2}\right]$$

Future Work

- Asymptotics for rest of cohomology groups

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- Asymptotics for rest of cohomology groups
- Full description of $\mathcal{A}_*^\chi$

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Mug and Torus morph

Thank you!
Questions?